DIFFERENTIAL GEOMETRY

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Contents

Summary

After the introduction of coordinates, it became possible to treat figures in plane and space by analytical methods, and calculus has been the main means for the study of curved figures. For example, one attaches the tangent line to a curve at each point. One sees how tangent lines change with points of the curve and gets an invariant called the curvature. C. F. Gauss, with whom differential geometry really began, systematically studied intrinsic geometry of surfaces in Euclidean space. Surface theory of Gauss with the discovery of non-Euclidean geometry motivated B. Riemann to introduce the concept of manifold that opened a huge world of diverse geometries. Current differential geometry mainly deals with the various geometric structures on manifolds and their relation to topological and differential structures of manifolds. Results in linear algebra (*Matrices, Vectors, Determinants and Linear Algebra*) and Euclidean geometry (*Basic Notions of Geometry and Euclidean Geometry*) are assumed to be known as aids in enhancing the understanding this chapter.

1. Curves in Euclidean Plane and Euclidean Space

Plane curves

A *curve* c in Euclidean plane $\mathbb{R}^2$ with orthogonal coordinates (x_1, x_2) is regarded as the locus of a moving point with time and given by a parametric representation

$$\mathbf{x}(t) = (x_1(t), x_2(t)),\ a \leq t \leq b. \tag{1}$$

It is assumed that functions $x_1(t), x_2(t)$ are of class C^k $(k \geq 2)$ and the tangent vector $\dot{\mathbf{x}}(t) = (\dot{x}_1(t), \dot{x}_2(t))$ to c is nonzero at each t. Then the *tangent line* to c at $\boldsymbol{x}(t_0)$ is given by $t \mapsto \mathbf{x}(t_0) + t\dot{\mathbf{x}}(t_0)$ in vector notation. The *arc length* $s(t)$ and the *length* $L(c)$ of c are defined as

$$s(t) = \int_a^t \|\dot{\mathbf{x}}(t)\| dt = \int_a^t \sqrt{\dot{x}_1^2(t) + \dot{x}_2^2(t)} dt \quad (a \leq t \leq b), \quad L(c) = \int_a^b \|\dot{\mathbf{x}}(t)\| dt. \tag{2}$$

Indeed, the length $L(c)$ doesn't depend on parameterizations of a curve. Since $s = s(t)$ is a strictly increasing function, one may take the inverse function $t = t(s)$ and gets a new parametric representation $s \to \mathbf{x}(s) = \mathbf{x}(t(s))$ of c by arc length s for which $\|\dot{\mathbf{x}}(s)\| \equiv 1$ holds. Setting $\mathbf{e}_1(s) = \dot{\mathbf{x}}(s)$ and $\mathbf{e}_2(s) = (-\dot{x}_2(s), \dot{x}_1(s))$, a unit *normal vector* to c given by rotating $\mathbf{e}_1(s)$ through 90° counterclockwise, one obtains a (positive) orthonormal basis $\{\mathbf{e}_1(s), \mathbf{e}_2(s)\}$ called the *Frenet frame* of c at each $\mathbf{x}(s)$.

Curvature

Let $c : \mathbf{x} = \mathbf{x}(s), 0 \leq s \leq L$ be parameterized by arc length. Then the acceleration vector $\ddot{\mathbf{x}}(s) = (\ddot{x}_1(s), \ddot{x}_2(s))$ of c is orthogonal to $\dot{\mathbf{x}}(s)$, and one may write $\ddot{\mathbf{x}}(s) = \dot{\mathbf{e}}_1(s) = \kappa(s)\mathbf{e}_2(s)$, where $\kappa(s)(= \kappa_c(s)) = \langle \ddot{\mathbf{x}}(s), \mathbf{e}_2(s) \rangle = \dot{x}_1(s)\ddot{x}_2(s) - \dot{x}_2(s)\ddot{x}_1(s)$ is called the *curvature* of c at $\mathbf{x}(s)$. Setting $\rho(s) = 1/\kappa(s)$ if $\kappa(s) \neq 0$, the circle centered at $\mathbf{x}(s) + \rho(s)\mathbf{e}_2(s)$ (center of curvature) of radius $|\rho(s)|$ is tangent to c of second order at $\mathbf{x}(s)$. The centers of curvature of c form a curve called the *evolute* of c. For a curve $c : \mathbf{x} = \mathbf{x}(t)$ parameterized by t, one obtains

$$\kappa(t) = \left\{ \dot{x}_1(t)\ddot{x}_2(t) - \ddot{x}_1(t)\dot{x}_2(t) \right\} \Big/ \left\{ \dot{x}_1(t)^2 + \dot{x}_2(t)^2 \right\}^{3/2}. \tag{3}$$

Here is an example: The locus of a fixed point on the circle of radius a rolling on x_1-axis is called the *cycloid* and given by $x_1(t) = a(t - \sin t), x_2(t) = a(1 - \cos t), 0 \leq t \leq 2\pi$. Its arc length and curvature are respectively $s(t) = 2a\{1 - \cos(t/2)\}$ and $\kappa(t) = -1/\{4a\sin(t/2)\}$. The evolute of a cycloid c defined for $-\infty < t < \infty$ is again a cycloid that is congruent to c.

What is the meaning of curvature? It appears in the *Frenet formula*:

$$\dot{\mathbf{x}}(s) = \mathbf{e}_1(s), \dot{\mathbf{e}}_1(s) = \kappa(s)\mathbf{e}_2(s), \dot{\mathbf{e}}_2(s) = -\kappa(s)\mathbf{e}_1(s) \tag{4}$$

that controls the local behavior of the Frenet frame and the curve itself. Now an angle $\theta(s) \in \mathbf{R}$ between the unit tangent vector $\dot{\mathbf{x}}(s)$ to c and a fixed unit vector may be defined so that $s \to \theta(s)$ is of class C^{k-1} and one has $\dot{\mathbf{x}}(s) = (\cos(\theta(s) + \alpha), \sin(\theta(s) + \alpha))$. Then $\kappa(s) = \dot{\theta}(s)$ holds, i.e. curvature is an intrinsic invariant of a curve. Indeed, curvature determines the curve: Suppose a C^{k-2}-

function $\kappa(s)$ is given on $[0,L]$. Then there exists a unique C^k-curve $c:\mathbf{x}=\mathbf{x}(s)$ in $\mathbb{R}^2$ parameterized by arc length with $\kappa_c(s)\equiv\kappa(s)$, up to parallel translations and rotations of $\mathbb{R}^2$. In the standard case of $\kappa(s)\equiv 0$ (resp. $\kappa(s)\equiv k\neq 0$), c is a (part of a) straight line (resp. circle of radius $1/|k|$). A curve given by the equation $\kappa(s)=s/a$ (a; a nonzero constant) is called a *clothoid* that describes the trajectory of a car running with unit speed and an increasing acceleration of the constant rate, and applied to the design of highways.

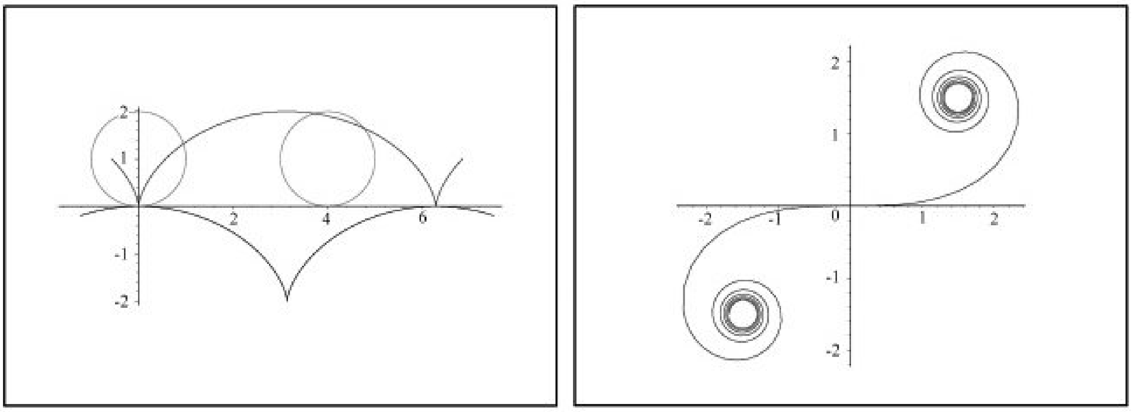

Figure 1. Cycloid and its evolute (left); clothoid (right)

Let $c:[0,L]\to\mathbb{R}^2$ be a *closed curve*, i.e. $\mathbf{x}^{(k)}(0)=\mathbf{x}^{(k)}(L)$ hold for k-th derivatives ($0\le k\le 2$). The integral $\int_0^L \kappa(s)ds$ is called the *total curvature* of c, and the *rotation number* of c is given by $(1/2\pi)\int_0^L \kappa(s)ds$ that turned out an integer. For a simple (i.e. without self-intersection points) closed curve c, the rotation number is equal to ± 1, and two closed curves are deformed to each other (by regular homotopy) if and only if they have the same rotation number. A simple closed curve c admits at least four vertices at which the derivative $\dot{\kappa}(s)$ vanishes.

Space curves

A curve c in Euclidean space $\mathbb{R}^3=\{(x_1,x_2,x_3)\,|\,x_i\in\mathbb{R}\}$ is given by

$$\mathbf{x}(t)=(x_1(t),x_2(t),x_3(t)),\ a\le t\le b \tag{5}$$

using a parametric representation. It is assumed that $x_i(t)\ (1\le i\le 3)$ are of class C^k ($k\ge 3$) and the acceleration vector $\ddot{\mathbf{x}}(t)=(\ddot{x}_1(t),\ddot{x}_2(t),\ddot{x}_3(t))$ is linearly independent of the tangent vector $\dot{\mathbf{x}}(t)=(\dot{x}_1(t),\dot{x}_2(t),\dot{x}_3(t))$. Arc length $s=s(t)$ is defined by (2) and one gets the parameterization of c by arc length. Then $\mathbf{e}_1(s)=\dot{\mathbf{x}}(s)$ is a unit vector and $\ddot{\mathbf{x}}(s)$ is orthogonal to $\dot{\mathbf{x}}(s)$. Now the *curvature* $\kappa(s)(=\kappa_c(s))$ of a space curve c is defined as $\|\ddot{\mathbf{x}}(s)\|$ that is assumed to be positive. Set $\mathbf{e}_2(s) = \ddot{\mathbf{x}}(s)/\kappa(s)$ and consider the vector product $\mathbf{e}_3(s) = \mathbf{e}_1(s) \times \mathbf{e}_2(s)$. One obtains a (positive) orthonormal basis $\{\mathbf{e}_i(s)\}_{1\le i\le 3}$ called the *Frenet frame* at each point $\mathbf{x}(s)$ of the curve. Then the following *Frenet-Serret formula*

$$\begin{cases} \dot{\mathbf{e}}_1(s) &= \kappa(s)\mathbf{e}_2(s) \\ \dot{\mathbf{e}}_2(s) &= -\kappa(s)\mathbf{e}_1(s) + \tau(s)\mathbf{e}_3(s) \\ \dot{\mathbf{e}}_3(s) &= -\tau(s)\mathbf{e}_2(s) \end{cases} \tag{6}$$

holds, where $\tau(s)(=\tau_c(s)) = \langle \dot{\mathbf{e}}_2(s), \mathbf{e}_3(s)\rangle = \det(\dot{\mathbf{x}}(s), \ddot{\mathbf{x}}(s), \dddot{\mathbf{x}}(s)) / \|\ddot{\mathbf{x}}(s)\|^2$ is called the *torsion* of c ($\det$ means the determinant of 3×3-matrix formed by the components of three vectors). The curvature and the torsion of a curve $\mathbf{x} = \mathbf{x}(t)$ parameterized by t are given by

$$\kappa(t) = \|\dot{\mathbf{x}}(t) \times \ddot{\mathbf{x}}(t)\| / \|\dot{\mathbf{x}}(t)\|^3, \quad \tau(t) = \det(\dot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t), \dddot{\mathbf{x}}(t)) / \|\dot{\mathbf{x}}(t) \times \ddot{\mathbf{x}}(t)\|^2. \tag{7}$$

Once $\kappa(s) > 0, \tau(s)\,(0 \le s \le L)$ are given, there exists a unique curve c parameterized by arc length with $\kappa_c(s) = \kappa(s), \tau_c(s) = \tau(s)$, up to parallel translations and rotations of $\mathbb{R}^3$. c is a plane curve if and only if its torsion vanishes everywhere, and a curve with constant curvature $\kappa(s) \equiv a > 0$ and constant torsion $\tau(s) \equiv b \ne 0$ is congruent to a regular *helix* given by

$$\mathbf{x}(s) = (a^2+b^2)^{-1}(a\cos(\sqrt{a^2+b^2}\,s), a\sin(\sqrt{a^2+b^2}\,s), b\sqrt{a^2+b^2}\,s).$$

For the total curvature $\kappa_c = \int_0^L \kappa(s)ds$ of a closed curve $c : \mathbf{x} = \mathbf{x}(s)\,(0 \le s \le L)$, $\kappa_c \ge 2\pi$ holds . Moreover if c is a knot, $\kappa_c > 4\pi$ holds .

2. Surfaces in Euclidean Space

Fundamental forms and curvature

In nature one sees various curved surfaces and nowadays curved surfaces are put use to designs, e.g. for cars. In differential geometry, one treats *parameterized surface* S given by a map

$$\mathbf{x} : D \ni (u_1, u_2) \mapsto \mathbf{x}(u_1, u_2) = (x_1(u_1,u_2), x_2(u_1,u_2), x_3(u_1,u_2)) \in \mathbb{R}^3 \tag{8}$$

from a domain D of u_1u_2-plane into $\mathbb{R}^3$, where $x_i(u_1,u_2)$ $(1 \le i \le 3)$ are functions of class C^k ($k \ge 2$). For example, the graph of a function $x_3 = f(x_1, x_2)$ is expressed as $(x_1, x_2) \to (x_1, x_2, f(x_1, x_2))$. The sphere $x_1^2 + x_2^2 + x_3^2 = r^2$ of radius r is given by $x_1 = r\cos u_1 \cos u_2$, $x_2 = r\cos u_1 \sin u_2$, $x_3 = r\sin u_1; -\pi/2 < u_1 < \pi/2, 0 \le u_2 \le 2\pi.$ To guarantee that S is a 2-dimensional figure, one takes curves $u_1 \mapsto \mathbf{x}(u_1, u_2)$, $u_2 \mapsto \mathbf{x}(u_1, u_2)$ on S fixing u_2, u_1 respectively and assumes that the tangent vectors to these parameter curves

$$\mathbf{x}_{u_i}(u_1, u_2) = \left(\frac{\partial x_1}{\partial u_i}(u_1, u_2), \frac{\partial x_2}{\partial u_i}(u_1, u_2), \frac{\partial x_3}{\partial u_i}(u_1, u_2) \right), (i = 1, 2)$$

are linearly independent and span the *tangent plane* T_pS to S at each $p = \mathbf{x}(u_1, u_2)$. Namely, the vector product $\mathbf{x}_{u_1}(u_1, u_2) \times \mathbf{x}_{u_2}(u_1, u_2) \neq 0$ everywhere, and one may take

$$\mathbf{e}(u_1, u_2) = \mathbf{x}_{u_1}(u_1, u_2) \times \mathbf{x}_{u_2}(u_1, u_2) \Big/ \left\| \mathbf{x}_{u_1}(u_1, u_2) \times \mathbf{x}_{u_2}(u_1, u_2) \right\|, \tag{9}$$

a unit *normal vector* to S. $\mathbf{x} : D \to \mathbb{R}^3$ is assumed to be injective on D.

Now the scalar product is induced on each T_pS. In terms of the first fundamental quantities

$$g_{ij} = \left\langle \mathbf{x}_{u_i}, \mathbf{x}_{u_j} \right\rangle \ (i, j = 1, 2;\ g_{12} = g_{21}) \tag{10}$$

it is given by $\left\langle \xi_1, \xi_2 \right\rangle = \sum_{i,j=1}^{2} g_{ij} a_i b_j$ for $\xi_1 = \sum_{i=1}^{2} a_i \mathbf{x}_{u_i}$, $\xi_2 = \sum_{i=1}^{2} b_i \mathbf{x}_{u_i} \in T_pS$. One may consider the norm of a vector and the angle between vectors in T_pS. For example, if a curve c on S is given by $[a, b] \ni t \mapsto \mathbf{x}(u_1(t), u_2(t))$, where $t \mapsto (u_1(t), u_2(t))$ is a C^k-curve in D, one has $\dot{\mathbf{x}}(t) = \sum_{i=1}^{2} \dot{u}_i(t) \mathbf{x}_{u_i} \in T_{\mathbf{x}(u_1(t), u_2(t))}S$. Then $\left\| \dot{\mathbf{x}}(t) \right\|^2 = \sum g_{ij}\, \dot{u}_i(t) \dot{u}_j(t)$ holds and the length $L(c)$ of c is given by (2). The scalar product on T_pS doesn't depend on parameterizations of the surface S, and a quadratic form $ds^2 = \sum_{i,j=1}^{2} g_{ij} du_i du_j$ on T_pS is called the *first fundamental form*. The *area* of S is given by

$$A(S) = \iint_D dS \ \text{ where } \ dS = \sqrt{\det(g_{ij})}\, du_1 du_2. \tag{11}$$

The *Gauss map* $G : S \to S^2$ is defined by assigning the unit normal vector $\mathbf{e}(u_1, u_2)$ to each point $p = \mathbf{x}(u_1, u_2)$ of S, where S^2 denotes the unit sphere in $\mathbb{R}^3$. To see how the unit normal vector $\mathbf{e}$ behaves on S, the second fundamental quantities are introduced by

$$h_{ij} = \left\langle \mathbf{x}_{u_i u_j}, \mathbf{e} \right\rangle = -\left\langle \mathbf{x}_{u_i}, \mathbf{e}_{u_j} \right\rangle = -\left\langle \mathbf{x}_{u_j}, \mathbf{e}_{u_i} \right\rangle \ (i, j = 1, 2;\ h_{12} = h_{21}). \tag{12}$$

Then for a fixed point $p = \mathbf{x}(u_1, u_2)$ of S, the signed distance from a nearby point

$\mathbf{x}(u_1+du_1, u_2+du_2)$ to the tangent plane T_pS is controlled by half of $II=\sum_{i,j=1}^{2} h_{ij}(u_1,u_2)du_i du_j$. II is called the *second fundamental form* of S, and is preserved under orientation preserving parameter transformations of S. If II is definite at p, i.e. its discriminant $D=4(h_{12}^2-h_{11}h_{22})<0$, S lies on a one side of T_pS around p where p is called an elliptic point. If $D=4(h_{12}^2-h_{11}h_{22})>0$ at p, S is located on both sides of T_pS where p is called a hyperbolic point.

Now to measure how S is curved in $\mathbb{R}^3$ around a point $p\in S$, take curves $c: s\mapsto \mathbf{x}(s)=\mathbf{x}(u_1(s),u_2(s))$ on S parameterized by arc length with $\mathbf{x}(0)=p$. For such a curve, the normal component of $\ddot{\mathbf{x}}(0)$, called the normal curvature, is given by

$$\kappa_n=\langle\ddot{\mathbf{x}}(0),\mathbf{e}\rangle=\sum_{i,j=1}^{2} h_{ij}\xi_i\xi_j \quad \text{with} \quad \dot{\mathbf{x}}(0)=\sum_{i=1}^{2}\xi_i\mathbf{x}_{u_i},\ \sum_{i,j=1}^{2} g_{ij}\xi_i\xi_j=1.$$

One considers the minimum and the maximum of normal curvatures $\sum_{i,j} h_{ij}\xi_i\xi_j$ at p under the condition $\sum_{i,j} g_{ij}\xi_i\xi_j=1$ that are called the *principal curvatures* of S at p. Now the *mean curvature* $H(=H_S)$ and the *Gaussian curvature* $K(=K_S)$ of S at p are defined as the arithmetic mean and the product of principal curvatures, respectively. They are given by

$$\begin{aligned} K &= \det(h_{ij})/\det(g_{ij}),\ H \\ &= (g_{11}h_{22}+g_{22}h_{11}-2g_{12}h_{12})/\{2\det(g_{ij})\}=\mathrm{tr}((g_{ij})^{-1}(h_{ij}))/2 \end{aligned} \tag{13}$$

where $\mathrm{tr}(a_{ij})=a_{11}+a_{22}$ is the trace. Gaussian curvature (resp. mean curvature) is invariant under (resp. orientation preserving) parameter transformations of S. For a plane one gets $K=H=0$, and for a sphere of radius r, $K=1/r^2, |H|=1/r$ hold. Note that if $K(p)>0$ (resp. <0), p is an elliptic (resp. hyperbolic) point. There are variety of flat (i.e. $K\equiv 0$) surfaces. Flat surfaces (with $H\neq 0$) obtained as a family of lines (ruled surfaces) are called *developable* surfaces including cones, circular cylinders, and tangent surfaces (consisting of tangent lines to a space curve). A *surface of revolution* given by rotating a curve $x_1=f(u_1), x_3=g(u_1)$ in x_1x_3-plane around x_3-axis is a typical surface for which K, H take simple forms.

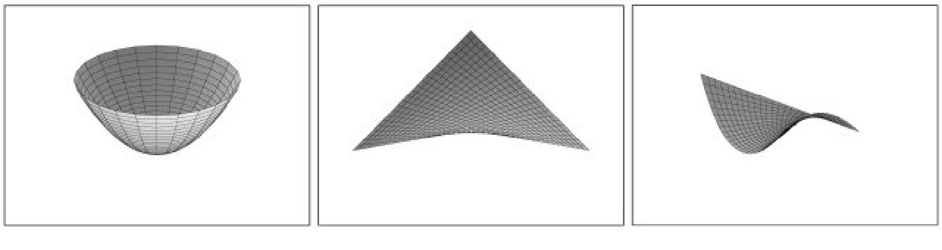

Figure 2. Gaussian curvature ($K>0$, $K<0$, $K\equiv 0$)

Now a surface S with $H \equiv 0$ is called a *minimal surface*, since such an S is stable with respect to the area. Belgian physicist J. Plateau verified experimentally that soap film obtained by dipping a wire form in soap solution has such a property. A surface of revolution given by $x_1 = \cosh x_3$ is a typical example of minimal surface called the *catenoid*. Minimal surfaces have been extensively studied in relation to partial differential equations and function theory of a complex variable: For example, if $f : \mathbb{R}^2 \to \mathbb{R}$ is a C^2-function whose graph is a minimal surface, then f is a linear function (Bernstein theorem); minimal surfaces are represented by holomorphic and meromorphic functions defined on unit open disk D (Weierstrass representation formula).

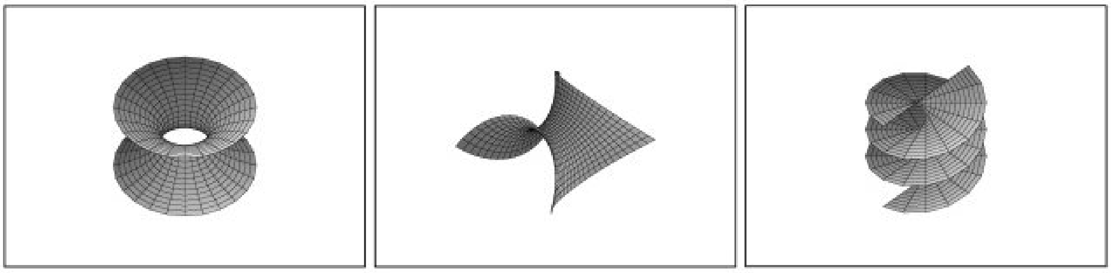

Figure 3. Minimal surfaces (catenoid, Enneper surface, right helicoid)

Intrinsic geometry of surfaces

It is possible to express the Gaussian curvature K in terms of the first fundamental quantities g_{ij} and their partial derivatives up to the second order, i.e. K is an intrinsic geometric invariant of S, as was shown and called “Theorema Egregium" by Gauss. The length of a curve on S is given by (2) in terms of g_{ij}, and then the intrinsic *distance* between two points on S is defined as the infimum of the lengths of curves on S joining them. Gauss developed his surface theory based on the distance where the Gaussian curvature plays an important role. Now, a curve $\gamma : \mathbf{x}(t) = \mathbf{x}(u_1(t), u_2(t))$ on S such that the orthogonal projection of the acceleration vector $\ddot{\mathbf{x}}(t)$ to $T_{\mathbf{x}(t)}S$ vanishes everywhere is called a *geodesic*. Then γ proceeds straight on S since it cannot feel any acceleration force on S, and satisfies

$$\ddot{u}_i(t) + \sum_{j,k=1}^{2} \Gamma^i_{jk} \dot{u}_j(t) \dot{u}_k(t) = 0 \quad (i = 1, 2)$$

where $\Gamma^k_{ij} = \sum_l g^{kl} (g_{il,j} + g_{jl,i} - g_{ij,l})/2; (g^{ij}) = (g_{ij})^{-1}$ and “$_{,k}$" denotes the differentiation with respect to u_k $(k = 1, 2)$. A geodesics γ exists at least for small $|t|$, once the initial point and direction are given, where $\|\dot{\gamma}(t)\|$ is constant. Now for a geodesic triangle Δ (a simply connected domain in S bounded by a closed curve consisting of three geodesic segments), A, B, C denote the (inner) angles of Δ. Then one obtains the *Gauss-Bonnet formula*:

$$A+B+C-\pi=\iint_{\Delta}KdS \tag{14}$$

If $K\equiv 0$ this is a familiar theorem that the sum of angles of a triangle in Euclidean plane is equal to π, and if $K>0$ (resp. $K<0$) on Δ the sum of angles of a triangle is greater (resp. less) than π. Indeed a surface S with $K\equiv -1$ gives a local model for hyperbolic geometry, although the whole hyperbolic plane cannot be realized as a surface in $\mathbb{R}^3$.

In the above one considered a piece of a surface, and now wants to define the whole surface. A subset S of $\mathbb{R}^3$ is called a *regular surface* of class C^k if the following holds: For each point $p\in S$ there exist an open neighborhood U of p in $\mathbb{R}^3$ and a parameterized C^k-surface $\mathbf{x}:D\to\mathbb{R}^3$ that is a homeomorphism from D onto $U\cap S$ with the relative topology. Thus each point of S admits a coordinate system (or *chart*) given by a homeomorphism $\mathbf{x}^{-1}:U\cap S\to D\subset\mathbb{R}^2$, and the whole S is described by an atlas consisting of such charts. For two charts $(U_i\cap S,\mathbf{x}_i^{-1})$ $(i=1,2)$ representing a point of S, the coordinate transformation is given by $\mathbf{x}_2^{-1}\circ\mathbf{x}_1:\mathbf{x}_1^{-1}(U_1\cap U_2)\to\mathbf{x}_2^{-1}(U_1\cap U_2)$ that is a C^k-map between domains of plane. A compact regular surface is called a closed regular surface (e.g. sphere, torus T that is the surface of a doughnut, and the surface Σ_k of a doughnut with k holes). Dividing a closed regular surface S into finitely many geodesic triangles and applying the Gauss-Bonnet formula, one obtains the *Gauss-Bonnet theorem* representing a topological invariant $\chi(S)$, the Euler characteristic of S, in terms of Gaussian curvature (note that $\chi(\Sigma_k)=2(1-k)$):

$$\chi(S)=\frac{1}{2\pi}\iint_S KdA. \tag{15}$$

Here are some global results that characterize the sphere: If the Gaussian curvature K of a closed surface is equal to a constant r, then $r>0$ and S is a sphere of radius $1/\sqrt{r}$. If the mean curvature K of a closed surface is equal to a constant h, then $h\neq 0$ and S is a sphere of radius $1/|h|$. A closed regular surface S in $\mathbb{R}^3$ admits an elliptic point. If $K>0$ everywhere, S is homeomorphic to sphere and called an *ovaloid*, since S lies on the one side of T_pS at each $p\in S$ and bounds a convex set of $\mathbb{R}^3$.

3. Differentiable Manifolds

Manifolds

The notion of *manifold* was introduced by B. Riemann in 1854 to set a foundation of geometry, i.e. how to grasp the concept of space that locally looks Euclidean space, but may spread manifold in curved manner and be of higher dimension. In the language of modern mathematics, manifold is defined as follows: Let M be a Hausdorff topological space. A pair (U,ϕ) of an open set U of M and a map $\phi:U\to\mathbb{R}^n$ is called a *chart* with coordinate neighborhood U, if ϕ is a homeomorphism onto an open subset of Euclidean space $\mathbb{R}^n$. A family of charts $\mathcal{A}=\{(U_\alpha,\phi_\alpha)\}_{\alpha\in A}$ is called an *atlas* of M if

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